Axisymmetric Relations in Linear Isotropic Elasticity and applications

From the book: Mechanics of Continuous Media: an Introduction

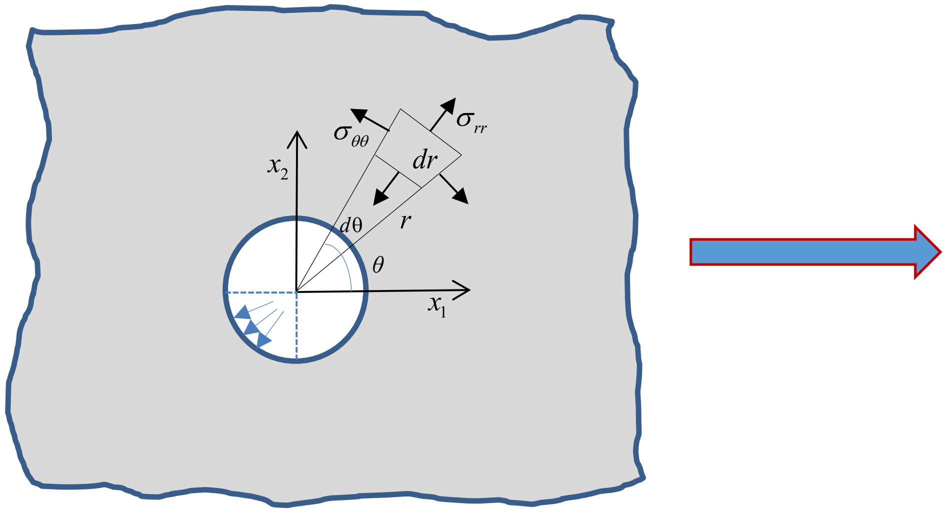
1. J Botsis and M Deville, PPUR 2018
2. J Botsis, Class notes given during the course

Mechanics of Solids: Axisymmetrically loaded members

STRUCTURAL COMPONENTS OF INTEREST

- Cylinders with internal and/or external pressures
- Rotating disks (annular or solid)
- ……….

Due to the rotational and axial symmetry the equations of elasticity are significantly simplified.



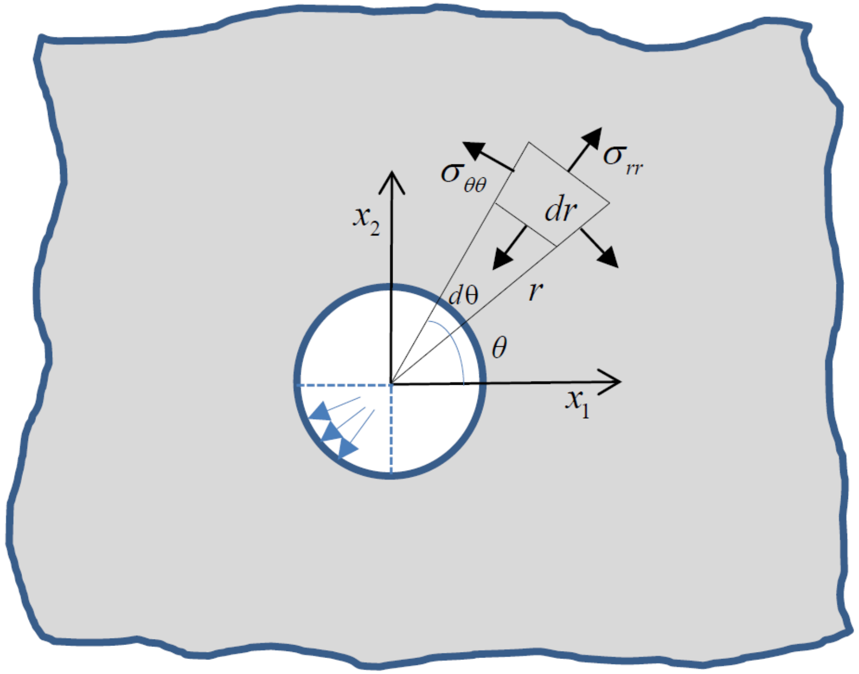
No axial stress $\Rightarrow \sigma_z = 0$

Symmetry around the z axis

1. deformation is independent of θ

2. $\sigma_{r\theta} = 0 \quad u_\theta = 0$

Navier Equations



With cylindrical symmetry the only Non-trivial Navier Equation is

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = 0$$

Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressures

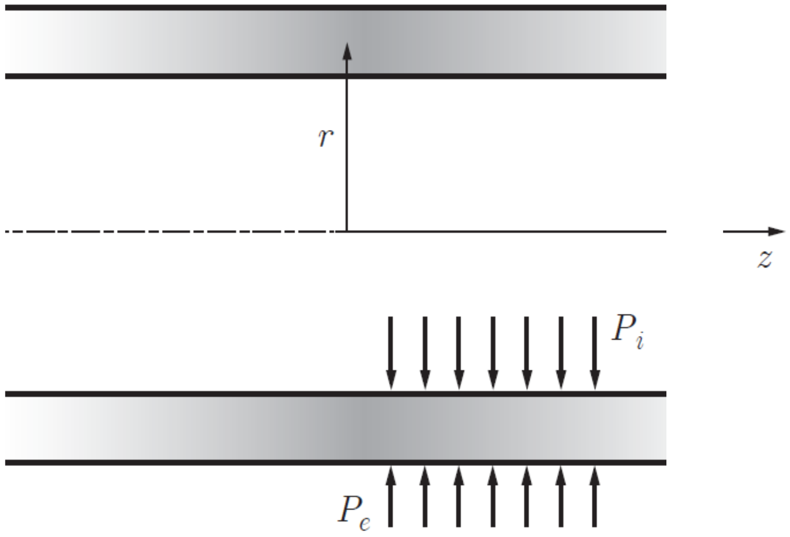
For a free ended cylinder under internal and external pressures we have a state of plane stress, independent of the axial coordinate z:

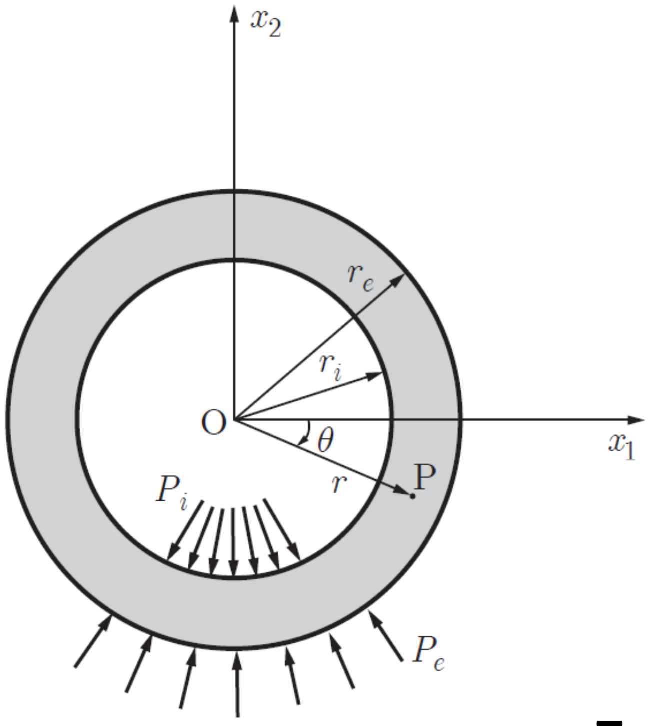
$$\Rightarrow \quad \sigma_{zz} = 0$$

$$\Rightarrow \quad \sigma_{r\theta} = \sigma_{rz} = 0$$

Due to the rotational symmetry, there is no tangential displacement:

$$\Rightarrow \quad u_\theta = 0$$





Parameters of the problem:

Stresses: σ_{rr}, $\sigma_{\theta\theta}$

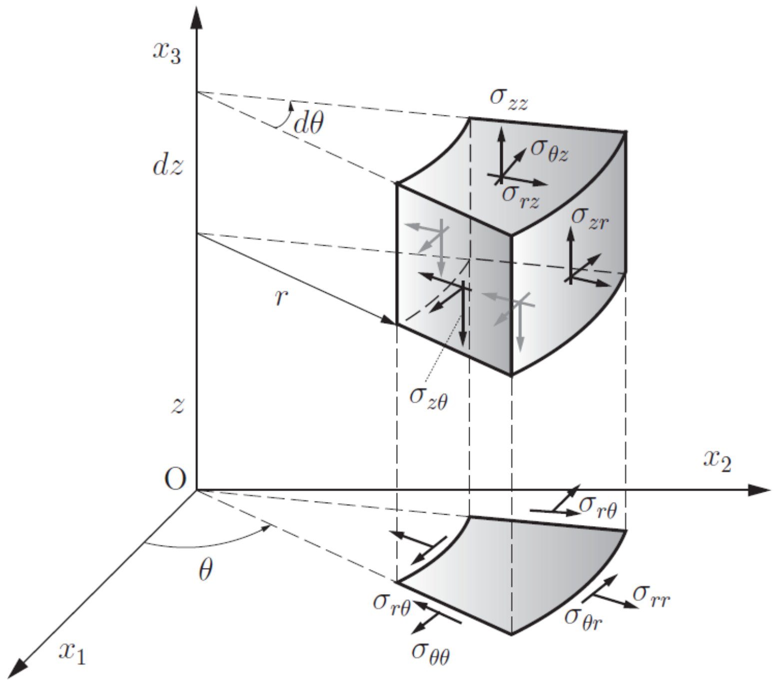
Strains: ε_{rr}, $\varepsilon_{\theta\theta}$

Displacement: u_r

THE STATE OF LOADING IS: PLANE STRESS

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Cylinder with internal and/or external pressure



With two stress components, σ_{rr}, σ_{rr}
the equilibrium equations become

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho b_\theta = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho b_z = 0 .$$

Assuming zero body forces,
the only remaining equation is:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

With only one non-zero displacement component
the strains are :

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

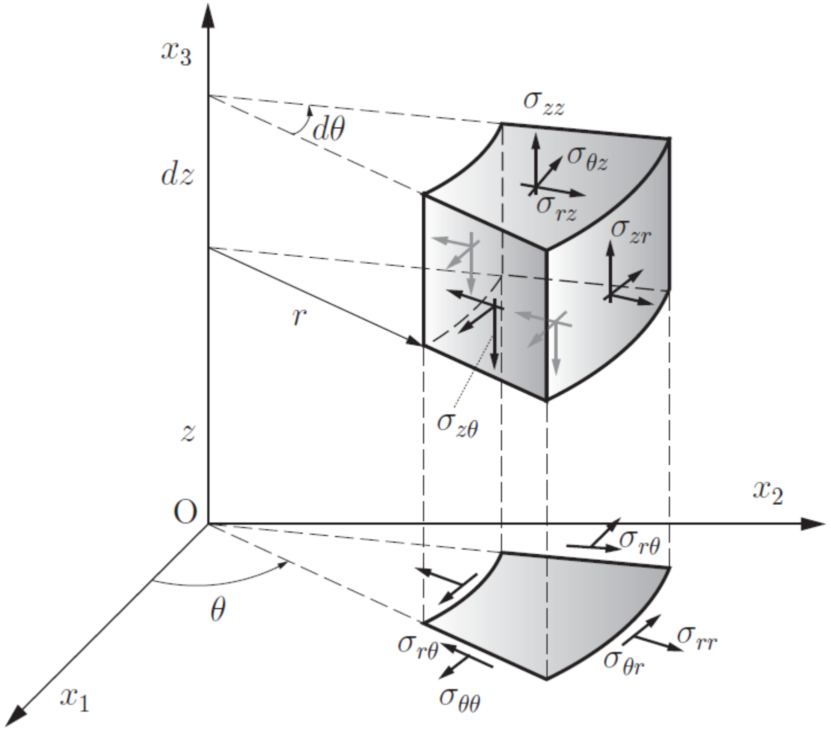
$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\varepsilon_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) .$$

$$\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Cylinder with internal and/or external pressure



STRESS-STRAIN RELATIONS (PLANE STRESS)
(we need only replace $x_1, x_2, x_3 \rightarrow r, \theta, z$ in the corresponding relations on Cartesian Coordinates)

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - v\sigma_{\theta\theta});$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - v\sigma_{rr});$$

$$\varepsilon_{r\theta} = \frac{1+v}{E}\sigma_{r\theta}$$

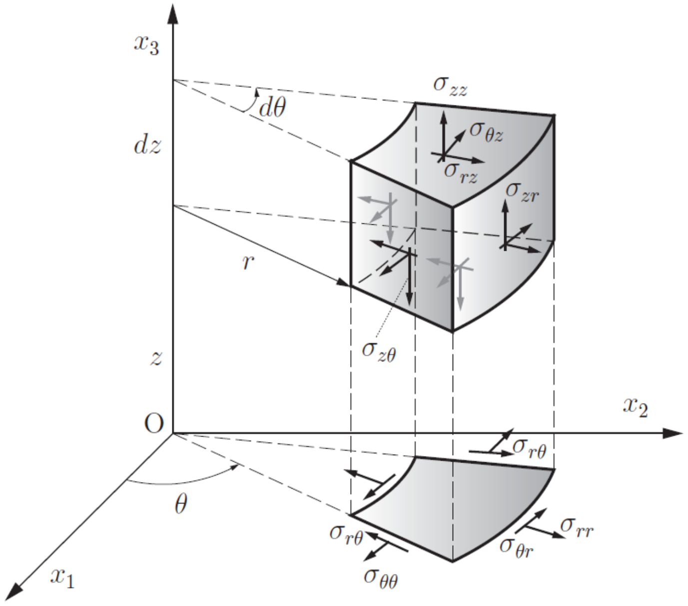
$$\sigma_{rr} = \frac{E}{1-v^2}(\varepsilon_{rr} + v\varepsilon_{\theta\theta});$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}(\varepsilon_{\theta\theta} + v\varepsilon_{rr});$$

$$\sigma_{r\theta} = \frac{E}{(1+v)}\varepsilon_{r\theta}$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Cylinder with internal and/or external pressure and free ends (plane stress state)



$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right)$$

$$\varepsilon_{rr} = \frac{du_r}{dr}; \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right)$$

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

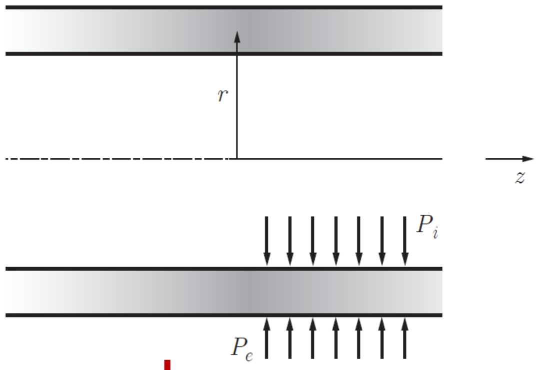
$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

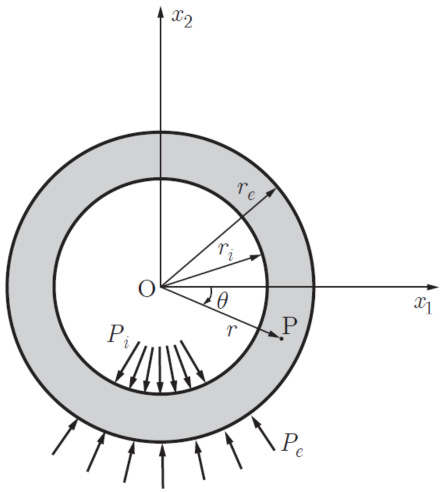
or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = 0$$

Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressure





$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right]=0$$

$$u_r = C_1 r + \frac{C_2}{r}\,.$$

$$\sigma_{rr}=\frac{E}{1-\nu^2}\left(\frac{du_r}{dr}+\nu\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta}=\frac{E}{1-\nu^2}\left(\frac{u_r}{r}+\nu\frac{du_r}{dr}\right)$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(C_1(1+\nu)-C_2\frac{1-\nu}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(C_1(1+\nu)+C_2\frac{1-\nu}{r^2}\right).$$

Boundary Conditions

$$\sigma_{rr} = -P_i,\quad \sigma_{r\theta}=0 \quad \text{at} \quad r=r_i$$

$$\sigma_{rr} = -P_e,\quad \sigma_{r\theta}=0 \quad \text{at} \quad r=r_e\,.$$

$$\sigma_{rr} = \frac{1}{r_e^2-r_i^2}\left(r_i^2P_i - r_e^2P_e + \frac{r_i^2r_e^2}{r^2}(P_e-P_i)\right)$$

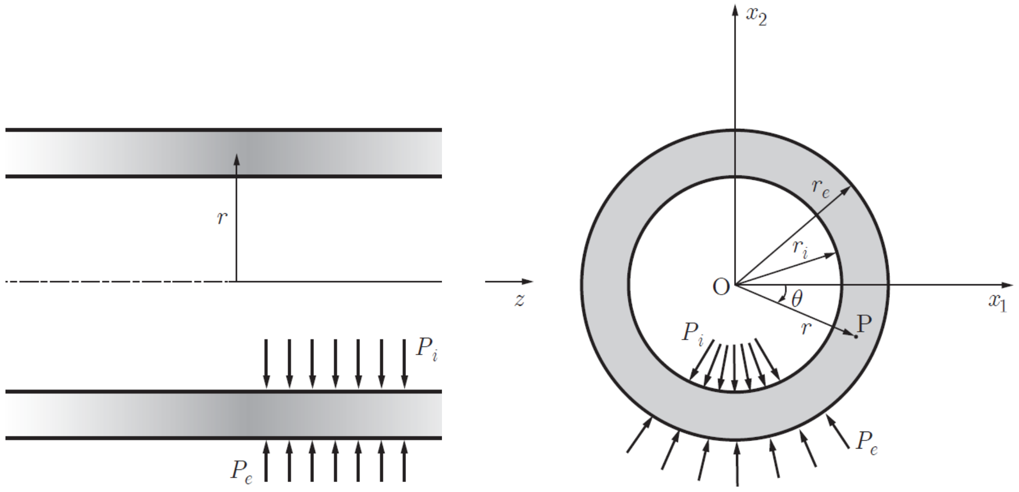
$$\sigma_{\theta\theta} = \frac{1}{r_e^2-r_i^2}\left(r_i^2P_i - r_e^2P_e - \frac{r_i^2r_e^2}{r^2}(P_e-P_i)\right)$$

$$u_r = \frac{1-\nu}{E}\,\frac{r_i^2P_i-r_e^2P_e}{r_e^2-r_i^2}\,r - \frac{1+\nu}{E}\,\frac{(P_e-P_i)}{r_e^2-r_i^2}\,\frac{r_i^2r_e^2}{r}\,.$$

Mechanics of Solids: Theory of Elasticity, Hollow Sphere under pressure

Approximation for Thin-Walled Container

Example: *Hollow Cylinder with Internal and External Pressures*



If the wall thickness is les than 10% of the inner radius, the cylinder is classified as a thin-walled.

The variation of stress with radius is disregarded and the following approximation can be adopted:

$$e = r_e - r_i \quad e \ll r_i$$

$$r_e^2 - r_i^2 = (r_e - r_i)(r_e + r_i) \approx 2er_i$$
$$r_i^2 P_i - r_e^2 P_e \approx r_i^2 (P_i - P_e)$$
$$r_e^2 \approx r_i^2 \qquad r^2 \approx r_i^2 .$$

$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$
$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{rr} \approx 0$$
$$\sigma_{\theta\theta} \approx \frac{r_i(P_i - P_e)}{e} .$$

Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressure

In plane stress we have a constant axial strain equal to,

$$\varepsilon_{zz} = -\frac{v}{E}\left(\sigma_{\theta\theta} + \sigma_{rr}\right)$$

with free ends we can write,

$$\boldsymbol{\sigma}_{zz} = v\left(\sigma_{\theta\theta} + \sigma_{rr}\right) + E\varepsilon_{zz} = c$$

The first term is due to Poisson effect and the second one due to the axial strain.

The total normal force at one of the cylinder's end is zero,

$$\int_{r_i}^{r_e} \sigma_{zz} 2\pi r \, dr = \pi c\left(r_e^2 - r_i^2\right) = 0 \Longrightarrow c = 0$$

Note that:

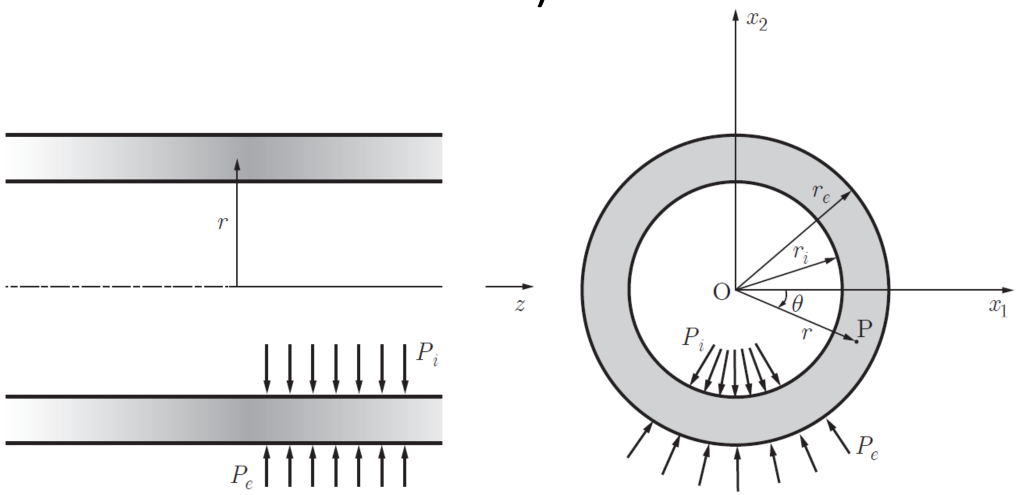
1. The stress components are the same as in the case of a cylinder subjected to internal and external pressures and fixed ends.

2. In such a case an axial stress is also present. (compare with the problem solved using potentials).

3. The radial displacement is different.

Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressure
(THICK-WALLED CYLINDER)



$$\sigma_{rr} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e + \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

$$\sigma_{\theta\theta} = \frac{1}{r_e^2 - r_i^2}\left(r_i^2 P_i - r_e^2 P_e - \frac{r_i^2 r_e^2}{r^2}(P_e - P_i)\right)$$

REMARK. We can use the same stress function for for both cases: cylinder with or without fixed ends.

The preceding solution of the problem was carried out using the Navier equation in terms of a single displacement component due to the axial and rotational Symmetries.

We can solve the same problem using the following Airy function, which is a general solution of $\nabla^4\Phi = 0$

$$\Phi(r) = A\ln r + Br^2 + Cr^2\ln r + D$$

An analysis of this equation, indicates that $C = D = 0$.
The non-zero stress components are

$$\sigma_{rr} = \frac{1}{r}\frac{d\Phi}{dr} = \frac{A}{r^2} + 2B \qquad \sigma_{\theta\theta} = \frac{d^2\Phi}{dr^2} = -\frac{A}{r^2} + 2B$$

Boundary Conditions

$$\sigma_{rr} = -P_i, \quad \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_i$$
$$\sigma_{rr} = -P_e, \quad \sigma_{r\theta} = 0 \quad \text{at} \quad r = r_e\ .$$

Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressure

SPECIAL CASES:

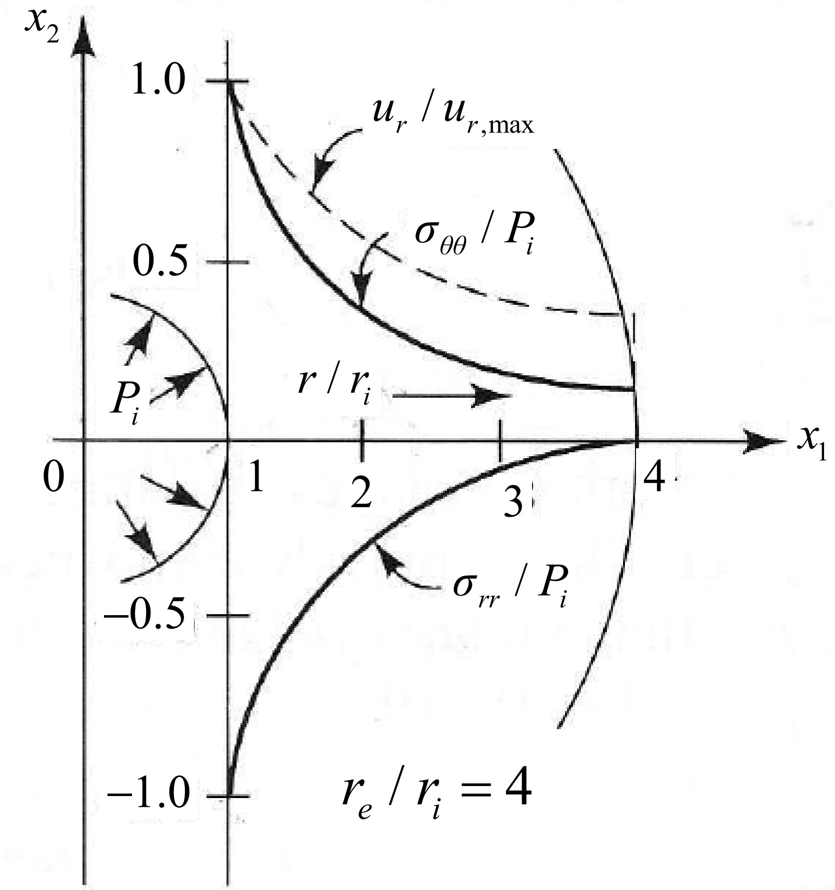
1: Internal Pressure only

The equations we obtained earlier reduce to:

$$\boldsymbol{\sigma}_{rr} = \frac{r_i^2 P_i}{r_e^2 - r_i^2}\left[1 - \frac{r_e^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{r_i^2 P_i}{r_e^2 - r_i^2}\left[1 + \frac{r_e^2}{r^2}\right]$$

$$u_r = \frac{r_i^2 P_i r}{E\left(r_e^2 - r_i^2\right)}\left[(1-v) + (1+v)\frac{r_e^2}{r^2}\right]$$



Cylinder with internal and/or external pressure

SPECIAL CASES:

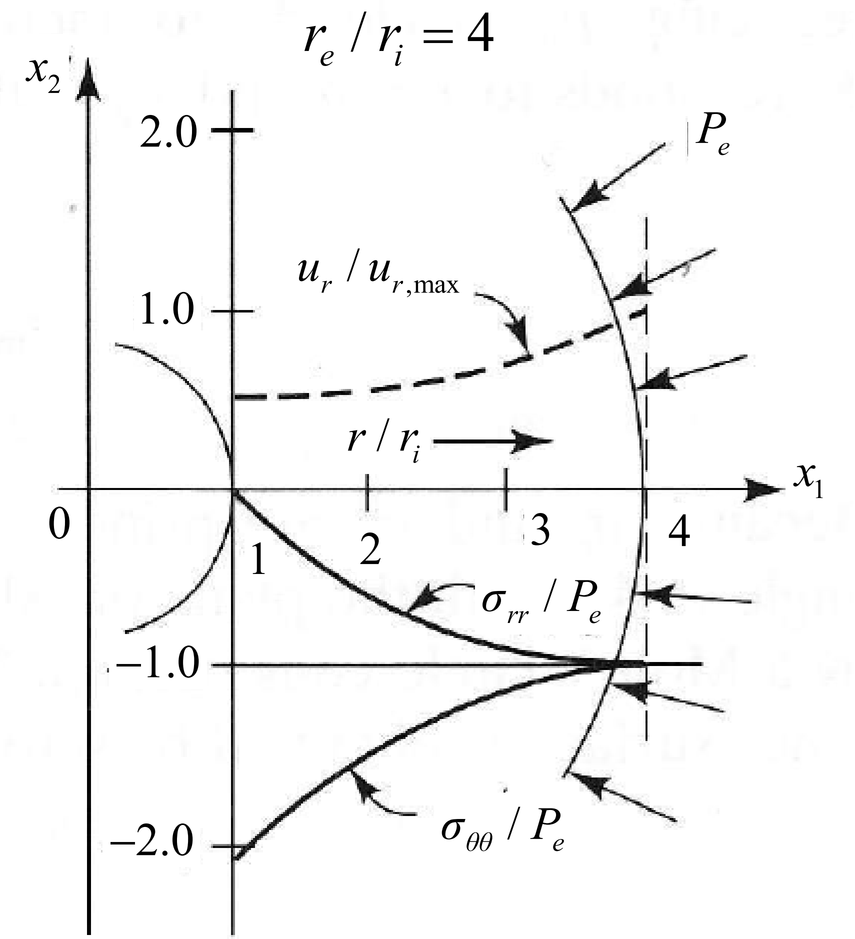
2: External pressure only

The equations we obtained earlier reduce to:

$$\sigma_{rr} = -\frac{r_e^2 P_e}{r_e^2 - r_i^2}\left[1 - \frac{r_i^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = -\frac{r_e^2 P_e}{r_e^2 - r_i^2}\left[1 + \frac{r_i^2}{r^2}\right]$$

$$u_r = -\frac{r_e^2 P_e r}{E\left(r_e^2 - r_i^2\right)}\left[(1-v) + (1+v)\frac{r_i^2}{r^2}\right]$$



Mechanics of Solids: Axisymmetrically loaded members

Cylinder with internal and/or external pressure

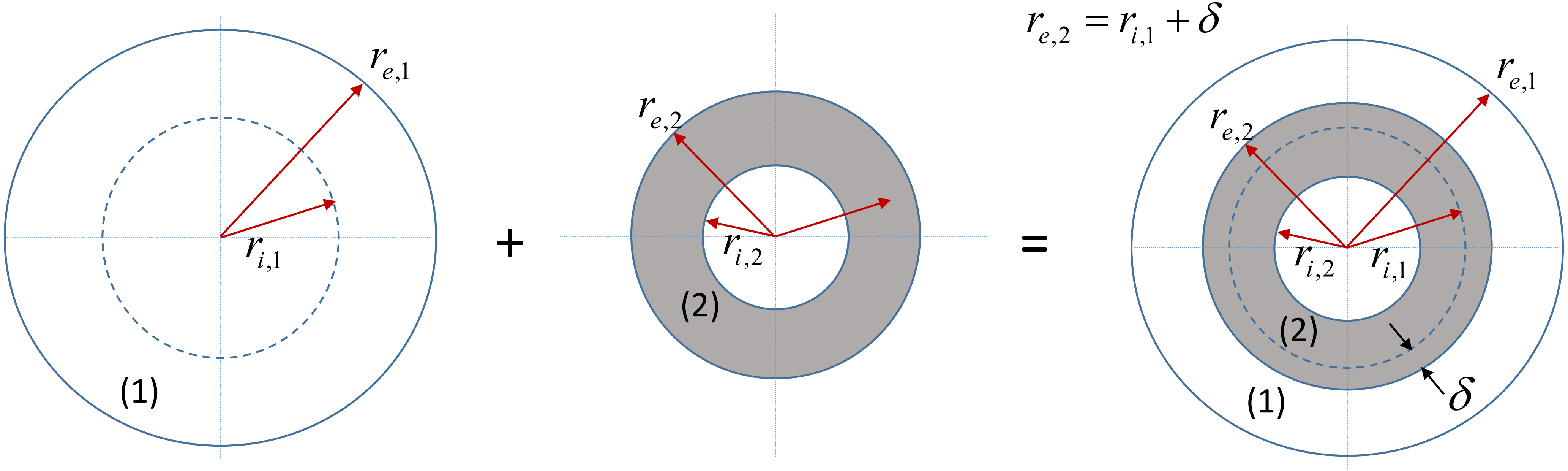
Application : Compound Cylinder

The internal radius of the outer cylinder (1) is smaller than the outer radius of the internal cylinder (2) by δ. The outer cylinder is heated and fit on to the inner one. After thermal equilibrium, pressure is developed around the contact surface. What is the developed pressure?

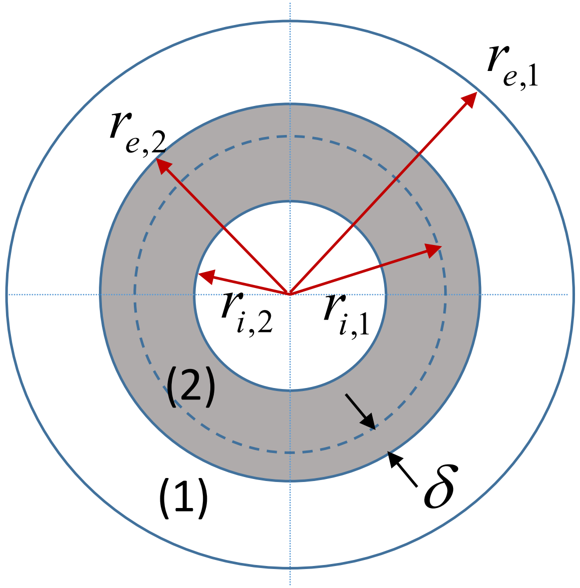
If properly designed, compound cylinders resist better relatively large pressures and require less material

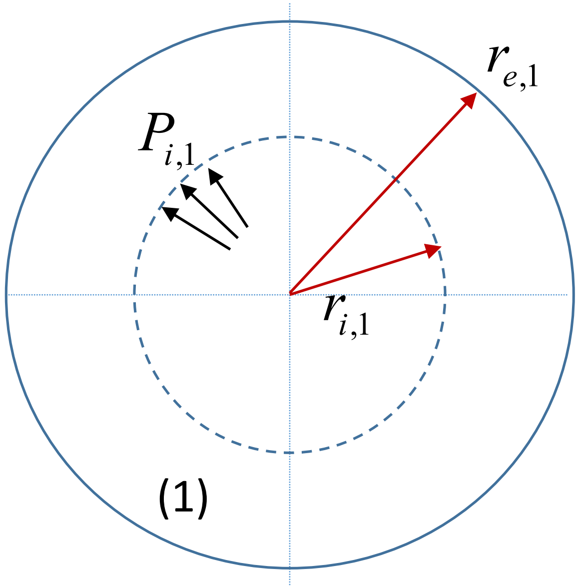
is called shrinking allowance

$$r_{e,2} = r_{i,1} + \delta$$



Mechanics of Solids: Axisymmetrically loaded members

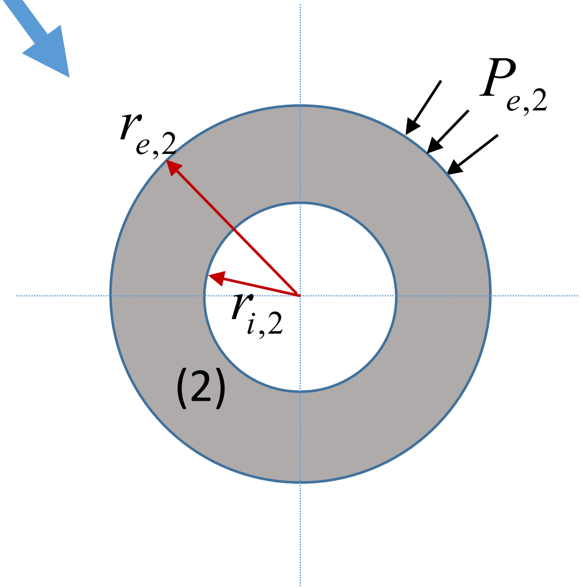




$$P_{i,1} = P; \quad P_{i,2} = 0$$

$$u_{r,1} = \frac{r_{i,1}^2 P r_{i,1}}{E\left(r_{e,1}^2 - r_{i,1}^2\right)}\left[\left(1-\nu\right)+\left(1+\nu\right)\frac{r_{e,1}^2}{r_{i,1}^2}\right]$$

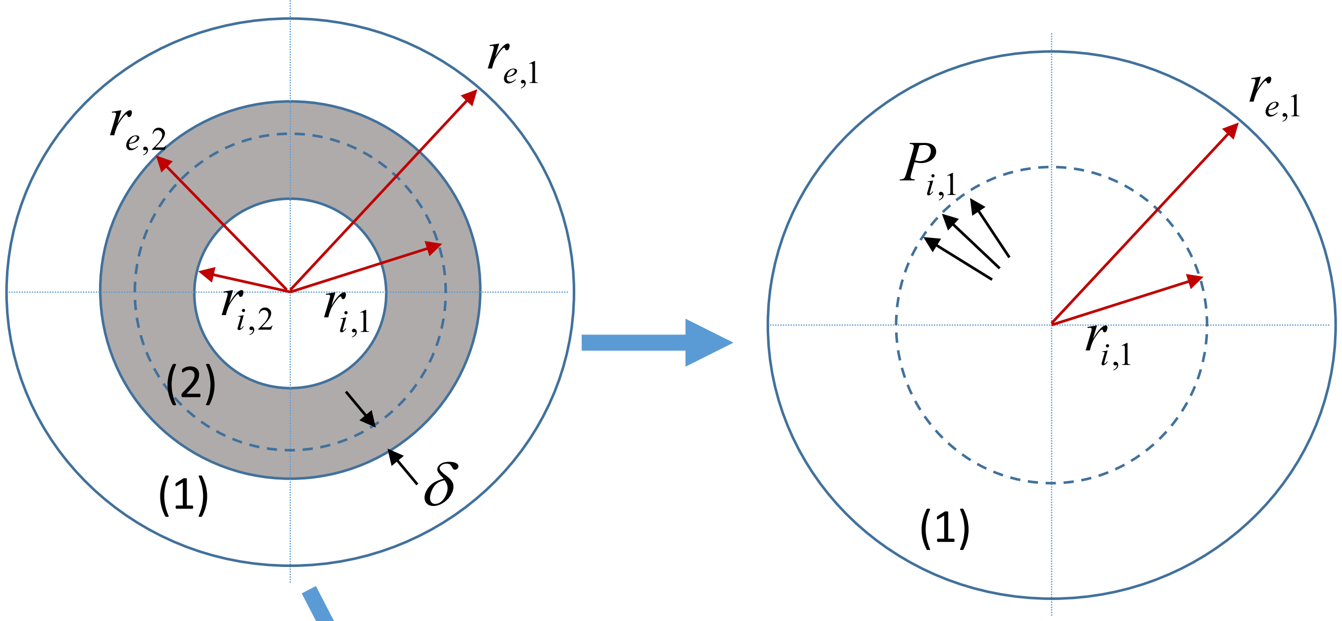
$$r_{e,2} = r_{i,1} + \delta$$

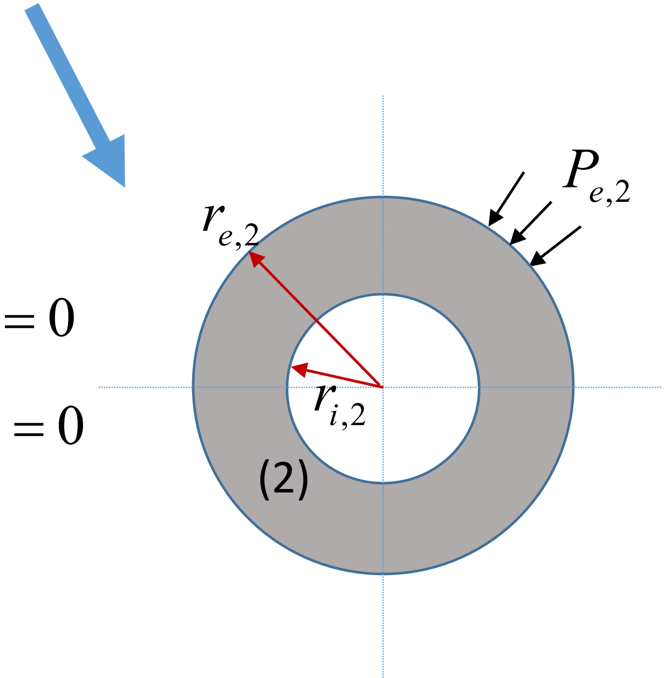


$$P_{e,2} = P; \quad P_{e,1} = 0$$

$$u_{r,2} = -\frac{r_{e,2}^2 P r_{e,2}}{E\left(r_{e,2}^2 - r_{i,2}^2\right)}\left[\left(1-\nu\right)+\left(1+\nu\right)\frac{r_{i,2}^2}{r_{e,2}^2}\right]$$

Mechanics of Solids: Axisymmetrically loaded members





$$P_{i,1} = P; \quad P_{i,2} = 0$$

$$P_{e,2} = P; \quad P_{e,1} = 0$$

$$r_{e,2} = r_{i,1} + \delta$$

For simplicity define:

$$r_{i,1} \approx r_{e,2} = b; \qquad r_{e,1} = c; \qquad r_{i,2} = a$$

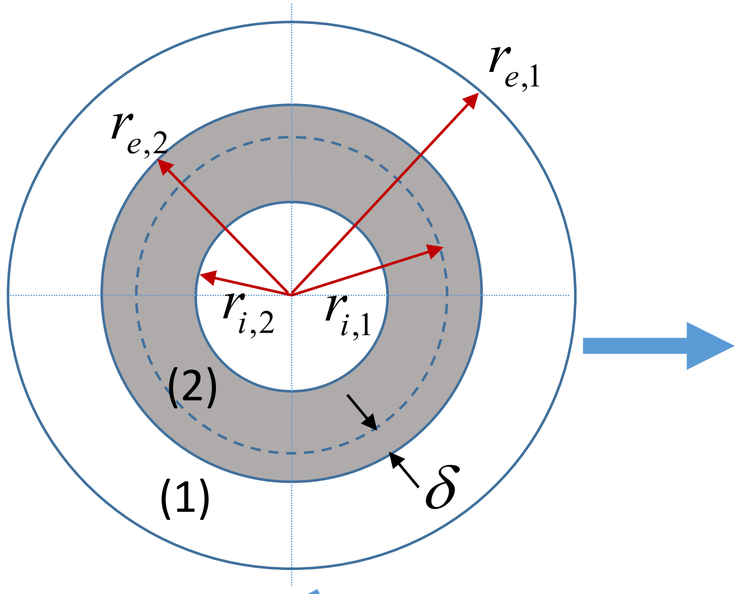
$$u_{r,1} = \frac{b^2 Pb}{E\left(c^2 - b^2\right)}\left[(1-\nu) + (1+\nu)\frac{c^2}{b^2}\right]$$

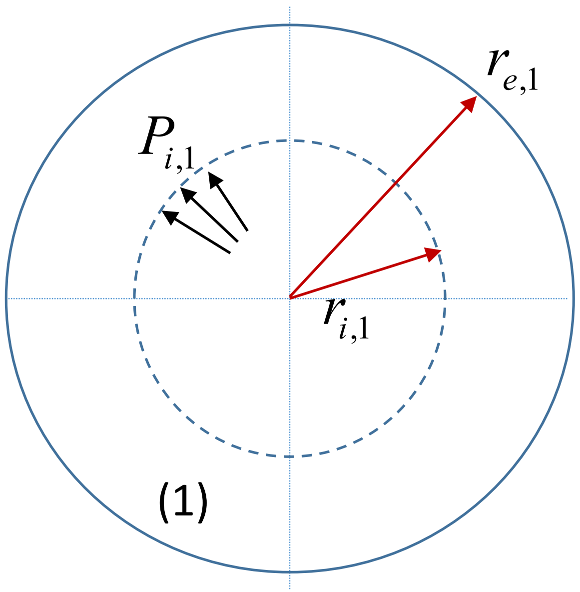
$$= \frac{Pb}{E}\left[\frac{c^2 + b^2}{c^2 - b^2} + \nu\right]$$

$$u_{r,2} = -\frac{b^2 Pb}{E\left(b^2 - a^2\right)}\left[(1+\nu) + (1-\nu)\frac{a^2}{b^2}\right]$$

$$= -\frac{Pb}{E}\left[\frac{b^2 + a^2}{b^2 - a^2} - \nu\right]$$

Mechanics of Solids: Axisymmetrically loaded members

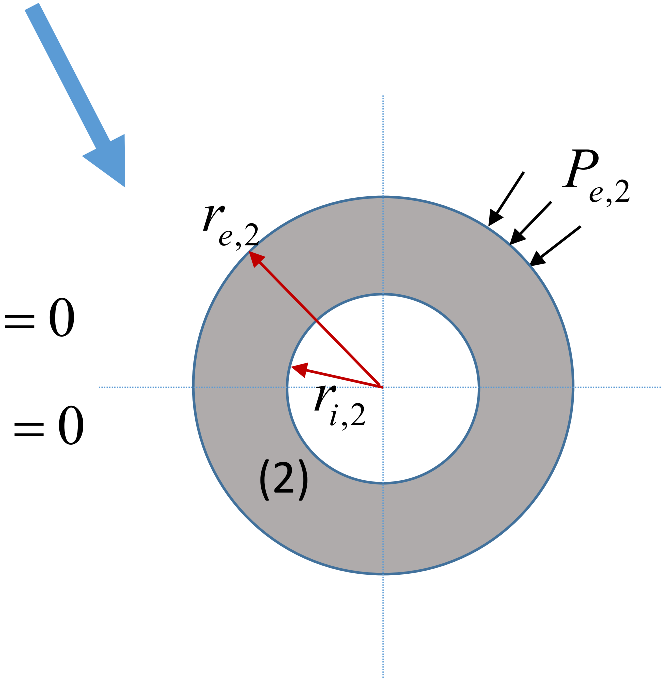




Compatibility:
The increase in the inner radius of cylinder (1), added to the decrease of the outer radius of cylinder (2), is equal to δ

$$\delta = \frac{Pb}{E}\left[\frac{c^2+b^2}{c^2-b^2}+v\right]+\frac{Pb}{E}\left[\frac{b^2+a^2}{b^2-a^2}-v\right]$$

$$P = \frac{E\delta}{b}\frac{(c^2-b^2)(b^2-a^2)}{2b^2(c^2-a^2)}$$



$$P_{i,1} = P; \quad P_{i,2} = 0$$

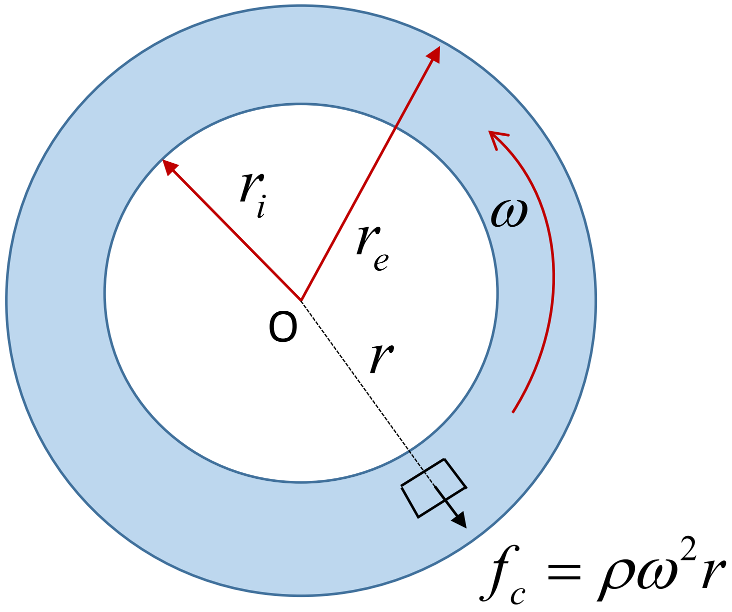
$$P_{e,2} = P; \quad P_{e,1} = 0$$

$$r_{e,2} = r_{e,1} + \delta$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Disks of constant thickness

(mass density ρ)



ω: angular speed in rad/sec.

We have here a cylindrical symmetry and all stresses are thickness independent.
The equilibrium equation is what we saw earlier with one more term, i.e., the centrifugal force:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0$$

Introduce in it the stresses in terms of displacements,

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right); \quad \sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right)$$

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = -(1-\nu^2)\frac{\rho\omega^2 r}{E}$$

$$u_r = u_{r,p} + u_{r,h}$$

$$u_r = -(1-\nu^2)\frac{\rho\omega^2 r^3}{8E} + c_1 r + \frac{c_2}{r}$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Disks of constant thickness

From the calculated displacement,

$$u_r = -(1-v^2)\frac{\rho\omega^2 r^3}{8E} + c_1 r + \frac{c_2}{r}$$

from stress-displacement relations,

$$\sigma_{rr} = \frac{E}{1-v^2}\left(\frac{du_r}{dr} + v\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\frac{u_r}{r} + v\frac{du_r}{dr}\right)$$

$$\sigma_{rr} = \frac{E}{1-v^2}\left[\frac{-(3+v)(1-v^2)\rho\omega^2 r^2}{8E} + (1+v)c_1 - (1-v)\frac{c_2}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left[\frac{-(1+3v)(1-v^2)\rho\omega^2 r^2}{8E} + (1+v)c_1 + (1-v)\frac{c_2}{r^2}\right]$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Annular Disks of constant thickness

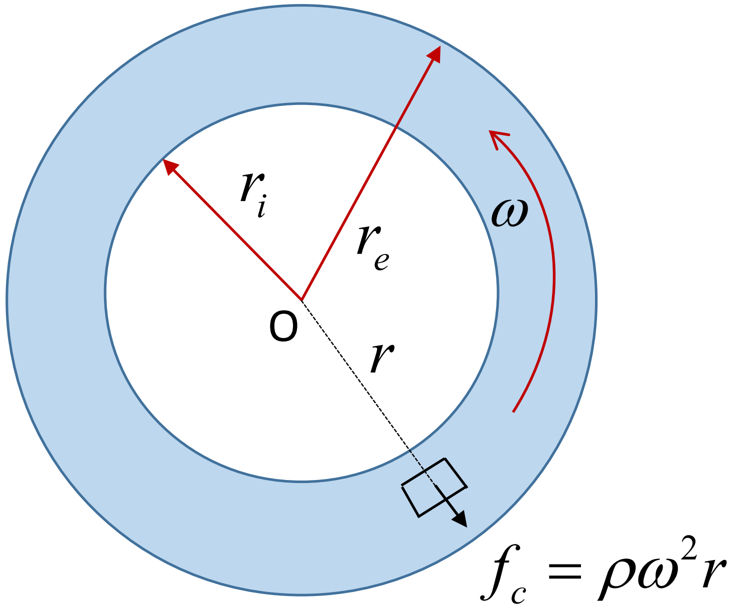
Stresses due to rotation without pressure

$$\sigma_{rr}\Big|_{r=r_i} = 0; \qquad \sigma_{rr}\Big|_{r=r_e} = 0$$

$$\sigma_{rr} = \frac{E}{1-v^2}\left[\frac{-(3+v)(1-v^2)\rho\omega^2 r^2}{8E} + (1+v)c_1 - (1-v)\frac{c_2}{r^2}\right]$$

$$0 = -\rho\omega^2 \frac{r_i^2}{E}\frac{(1-v^2)(3+v)}{8} + (1+v)c_1 - (1-v)\frac{c_2}{r_i^2}$$

$$0 = -\rho\omega^2 \frac{r_e^2}{E}\frac{(1-v^2)(3+v)}{8} + (1+v)c_1 - (1-v)\frac{c_2}{r_e^2}$$



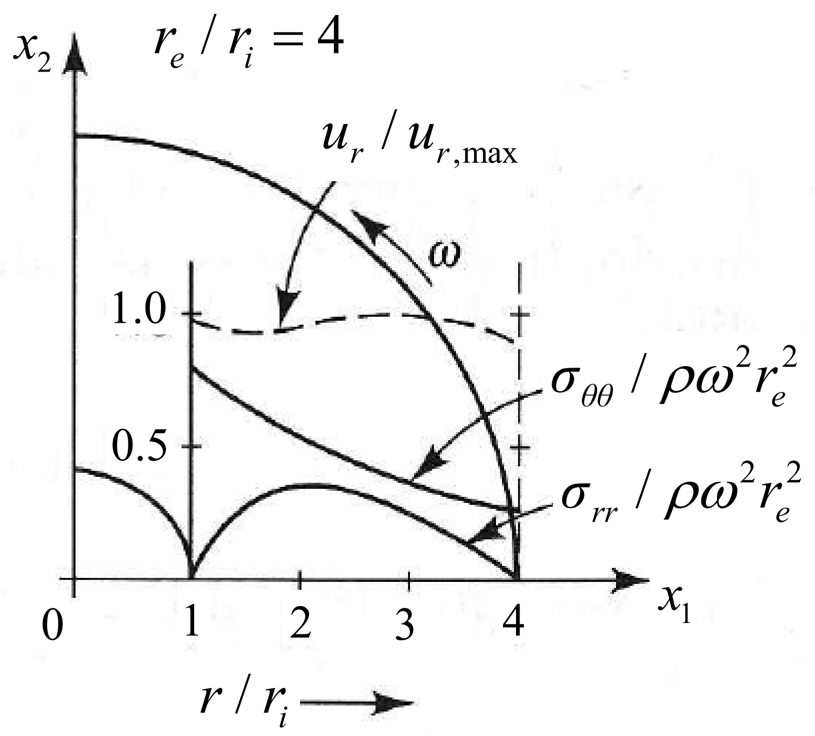
$$c_1 = \rho\omega^2 \frac{r_i^2 + r_e^2}{E}\frac{(1-v)(3+v)}{8}$$

$$c_2 = \rho\omega^2 \frac{r_i^2 r_e^2}{E}\frac{(1+v)(3+v)}{8}$$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Disks of constant thickness

Stresses due to rotation without pressure,



$$\sigma_{rr} = \frac{(3+v)}{8}\left(r_i^2 + r_e^2 - r^2 - \frac{r_i^2 r_e^2}{r^2} \right)\rho\omega^2$$

$$\sigma_{\theta\theta} = \frac{(3+v)}{8}\left(r_i^2 + r_e^2 - \frac{1+3v}{3+v} r^2 + \frac{r_i^2 r_e^2}{r^2} \right)\rho\omega^2$$

$$u_r = \frac{(3+v)(1-v)}{8E}\left(r_i^2 + r_e^2 - \frac{1+v}{3+v} r^2 + \frac{1+v}{1-v}\frac{r_i^2 r_e^2}{r^2} \right)\rho\omega^2 r$$

Taking $\dfrac{d\sigma_{rr}}{dr} = 0$

we see **maximum** radial stress at $r = \left(r_i r_e \right)^{1/2}$

Mechanics of Solids: Theory of Elasticity; Eqs in cylindrical coordinates

Rotating Solid Disks of constant thickness

Boundary conditions:

$$r_i = 0; \quad \sigma_{rr}\big|_{r=r_b} = 0; \quad u_r\big|_{r=0} = 0$$

$$c_1 = \rho\omega^2 \frac{r_e^2}{E}\frac{(1-\nu)(3+\nu)}{8}; \quad c_2 = 0$$

$$\sigma_{rr} = \frac{3+\nu}{8}\left(r_e^2 - r^2\right)\rho\omega^2; \quad \sigma_{\theta\theta} = \frac{3+\nu}{8}\left(r_e^2 - \frac{1+3\nu}{3+\nu}r^2\right)\rho\omega^2$$

$$u_r = \frac{(1-\nu)}{8E}\left((3+\nu)r_e^2 - (1+\nu)r^2\right)\rho\omega^2 r$$